Stability of projective Varieties

- Let G be reductive algebraic group over $\mathbb{C}$. $((\mathbb{C}^*)^n, SL(N,\mathbb{C}))$
- V be an n-dim representation of G
- $x \in V$, f is G invariant homogeneous polynomial

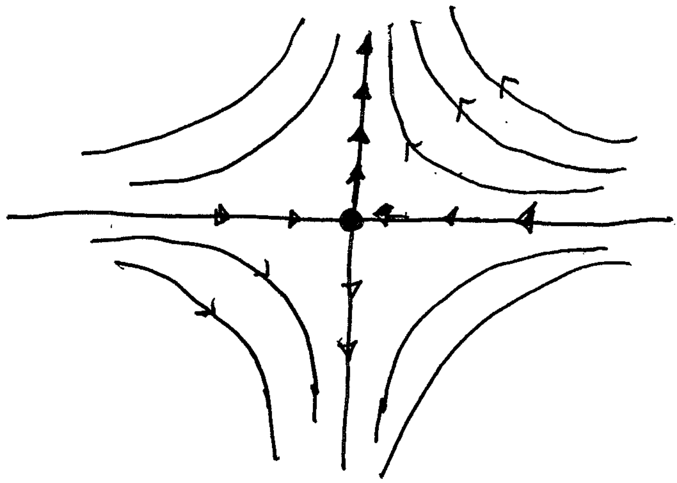
Stability Chart

	x unstable	x semi stable	x stable
A	$0 \in \overline{G \cdot x}$	$0 \notin \overline{G \cdot x}$	(i) $G \cdot x$ is closed (ii) $Stab(x)$ is finite
B	$\forall f$, $f(x)=0$	$\exists f$ s.t. $f(x) \neq 0$	(i) $\forall y \in V - G \cdot x$ $\exists f$ s.t. $f(x) \neq f(y)$ (ii) $Stab(x)$ is finite
C	$\exists$ a 1-PS λ of G s.t. the weights of x are all positive (or all negative)	$\forall$ 1-PS λ of G weights are not all positive (or all negative)	$\forall$ nontrivial λ x has positive and negative weights

(A is the definition, B allows for construction of quotient (at least for me), C is easiest for practice? examples?)

$t \in \mathbb{C}^*$, $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ acting on $\mathbb{C}^2$

(no arrows)



(orbits, $t \to 0$)

(for $\alpha \in \mathbb{C}^*$ orbits are $xy = \alpha$)

[$\boxed{\text{Test}}$ for A we can see which orbits are closed, which contain 0]

for C, let (x_0, y_0), ~~[illegible]~~ $x_0, y_0 \neq 0$

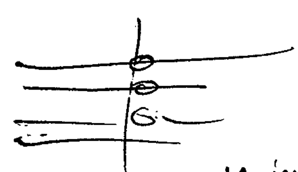
the weights are $(1, -1)$ however

look at $x = 0$, weights are -1

(P) ring of invariants $\mathbb{C}[x,y]^{\mathbb{C}^*} = \mathbb{C}[xy]$

$\bigoplus_{m \in \mathbb{N}} (xy)^m \leftarrow$ vanishes on unstable orbits

away from unstable orbits separates points.



y invariant polynomial

$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$

Note C is called the Hilbert-Mumford criterion

Projective varieties

on $\mathbb{C}^{n+1}$, $\mathbb{C}^* \circlearrowright \mathbb{C}[x_0, \ldots, x_n]$

eigen vectors for this action are homogeneous polynomials.

so $\tilde{X} \triangleq (p_1 = \cdots = p_K = 0)$ in $\mathbb{C}^{n+1}$ is an affine variety

defines $X \subseteq \mathbb{P}^n$ projective variety

can also work on $\mathbb{P}^n$ by thinking of $p_K \in H^0(\mathbb{P}^n, \mathcal{O}(\deg p_K))$

$G \subseteq SL(n+1, \mathbb{C})$ acts on $\mathbb{P}^n$

$$T \bullet [x_0 : \cdots : x_n] = [T_{0j} x_j : \cdots : T_{nj} x_j]$$

thus $SL(n+1, \mathbb{C})$ acts on $X \subseteq \mathbb{P}^n$

$\boxed{\text{GIT}}$ Allows us to construct X/G

for $r \gg$, X/G is the image of X under the following embedding

$$X \longrightarrow \mathbb{P}\left(\left(H^0(X, \mathcal{O}(r))^G\right)^* \right)$$

$$x \longmapsto [s_0(x) : \cdots : s_K(x)] \in \mathbb{P}^K$$

where s_i form a basis for $H^0(X, \mathcal{O}(r))^G$
(G-invariant guys)

Key, use Aubin–Yau energy as a "moment map"

$$F_\omega^0(\varphi) = -\frac{1}{V}\int_0^1 \int_X \dot{\varphi}(t)\, \omega_\varphi^n$$

$$= J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\, \omega^n$$

Fix $\omega = i\partial\bar\partial \log |X|^2 \leftarrow$ FS

Let $\varphi_\sigma = \log\left(\frac{|\sigma X|^2}{|X|^2}\right) = \cancel{\log\left(\frac{X^t \bar\sigma^t \sigma X}{|X|^2}\right)} = \log\left(\frac{X^* \sigma^* \sigma X}{X^* X}\right)$

So $\sigma^* \omega = \omega + i\partial\bar\partial\varphi_\sigma \triangleq \omega_\sigma$

Let C be traceless hermitian matrix, $\sigma(t) = e^{Ct}\sigma_0$

σ_0 is critical point of F_ω^0 iff

$$\left.\frac{d}{dt}\right|_{t=0} F_\omega^0(\varphi_{\sigma(t)}) = 0$$

Note $\dot{\varphi}_{\sigma(t)} = \frac{X^* \sigma^* (C^* + C)\sigma X}{X^* \sigma^* \sigma X}$

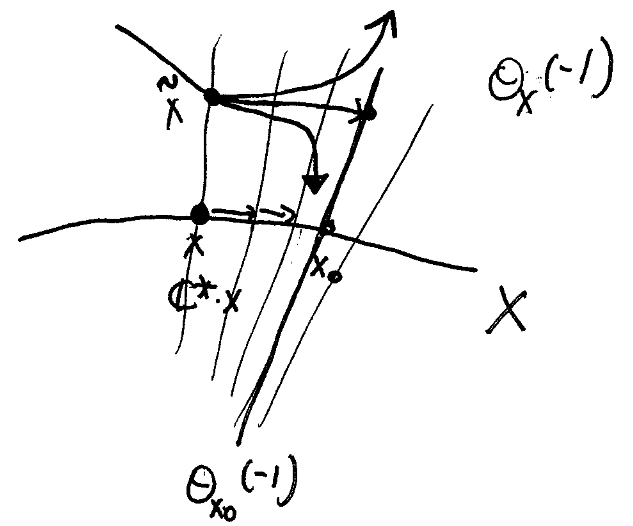
Lemma

(a) $\left.\frac{d}{dt}\right|_{t=0} F_\omega^0(\varphi_{\sigma(t)}) = -\frac{1}{V}\int_X \frac{X^* \sigma^* (C^* + C)\sigma X}{X^* \sigma^* \sigma X}\, \omega_\varphi^n$

Map only defined on semi-stable points (by B)

however, it may contract more than just G-orbits if the orbits are not stable

$\boxed{\mathbb{C}}$ on projective varieties



$$x_0 = \lim_{\lambda \to 0} \lambda \cdot x$$

Lets do some examples

n points in $\mathbb{P}^1$ (with multiplicities)

degree n homogeneous polynomials in $\mathbb{C}^2$

$$f = a_1 x^n + a_2 x^{n-1} y + a_3 x^{n-2} y^2 + \cdots + a_{n+1} y^n$$

We can consider 1-PS of $SL(2, \mathbb{C})$

$$\lambda(t) = \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix}$$

$$\lambda(t)\cdot f \quad \begin{pmatrix} t^{kn} & & & & \\ & t^{k(n-2)} & & & \\ & & t^{k(n-4)} & & \\ & & & \ddots & \\ & & & & t^{-kn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

(If a_i is zero we write the cooresponding entry as zero)

now $\lambda f \to \infty$ as $t \to 0$

as long as there is at least 1 negative power of t
(problem is this depends on coordinates)

or, if $X=0$, then $f\big|_{X=0}$ does not vanish to order $\geq \frac{n}{2}$

this can be made coordinate free

(this is awesome wish we had it)

by saying f is (semi)stable $\iff$ for all points f does not vanish $(\geq) > \frac{n}{2}$ at nonzero point

$a_1\ a_2 \ldots \big| \ldots, a_{n+1}$ $\frac{n}{2}$

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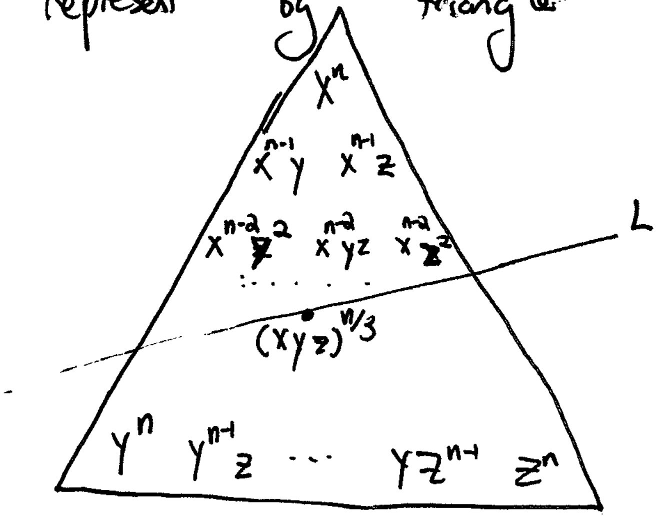
let $n = 2$

x^2	$\leftarrow$ unstable	$\begin{pmatrix} t^{2k} & & \\ & 0 & \\ & & 0 \end{pmatrix}$	limit is 0
$x^2 + xy$	$\leftarrow$ semi-stable	$\begin{pmatrix} t^{2k} & & \\ & t^0 & \\ & & 0 \end{pmatrix}$	limit is xy
$x^2 + xy + y^2$	$\leftarrow$ stable	$\begin{pmatrix} t^{2k} & & \\ & t^0 & \\ & & t^{-2k} \end{pmatrix}$	limit is ∞

Cool way to think of stability (before we had the vanishing coord free condition)

f is a homo polynomial degree n

represent by triangle



coordinatize triangle by

i_x, i_y, i_z

(exponents of x, y, z)

the condition that L pass through center of triangle

$$L = \{ a i_x + b i_y + c i_z = 0 \}$$

$(a,b,c) \neq (0,0,0)$

is that $a+b+c=0$

$G = SL(3, \mathbb{C})$

consider 1-PS $\lambda(t) \longmapsto \begin{pmatrix} t^a & & \\ & t^b & \\ & & t^c \end{pmatrix}$ (can do for all 1-PS subgroups)

the weights of this action $\lambda(t) f$ are values of form defining L at non-zero coefficients

f unstable $\Leftrightarrow$ in some coordinates, all nonzero coefficients lie to one side of some L

f semistable $\Leftrightarrow$ $\forall$ coordinates, f has non-zero coefficients on boths sides of L or on L

f -stable $\Leftrightarrow$ $\forall$ coordinates, f has non-zero coeff. on both sides of L

Condition B

If you like if $a_1x^2 + a_2xy + a_3y^2$

coordinates $(a_1, a_2, a_3) \in \mathbb{C}^3$

invariant polynomials in $\mathbb{C}[a_1, a_2, a_3]$ are generated by a_2 and a_1a_3

thus a_2xy <u>not</u> in orbit of $a_1x^2 + a_2xy$, yet no invariant polynomial can tell these points apart.

Anyways.... we want to talk about stability of projective varieties, not points in projective varieties.

<u>Example</u> Curves in $\mathbb{P}^2$

(still thought of as points in some vector space)

homo. ~~symmetric~~ polynomials on $\mathbb{C}^3$

say degree 2 to make our lives easier

$$\begin{array}{ccc} & x^2 & \\ xy & & xz \\ y^2 & yz & z^2 \end{array}$$

let $f = a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2$

$$\begin{array}{ccc} & a_1 & \\ a_2 & & a_3 \\ a_4 & a_5 & a_6 \end{array}$$

next page first

degree 2

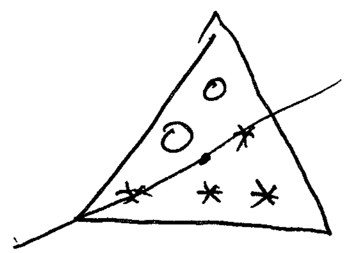
$$\begin{array}{ccc} & a_1 & \\ & a_2 \quad a_3 & \\ a_4 & a_5 & a_6 \end{array}$$

$$f = a_1 x^2 + a_2 xy + a_3 xz + a_4 y^2 + a_5 yz + a_6 z^2$$

Choose coord so $(1,0,0) \in f$

and $z=0$ is tangent at that point

$\Rightarrow \quad a_1 = a_2 = 0$



at the very least

Never stable

yet smoothness can imply stability

the derivative of f is $\begin{pmatrix} a_3 z \\ 2a_4 y + a_5 z \\ a_3 x + a_5 y + 2a_6 z \end{pmatrix}$

smoothness $\Rightarrow \quad a_3 \neq 0$ — look at derivative at $(1,0,0)$

[$a_4 \neq 0$ as well] (or else you can factor out a z and get intersection of 2 planes)

[look up discriminant]

We'll come back to this

thus we have seen for hypersurfaces in $\mathbb{P}^n$ it makes sense to talk about stability, ~~since~~ for the coefficients of the polynomial ~~define~~ give coordinates in some big $\mathbb{C}^N$, can talk about stability of those points

for a general $X \subset \mathbb{P}^n$, what do we mean by X is stable?

<u>Chow Stability</u> suppose $\dim X = m$, $\deg(X) = d$

let Z be the set of $(n-m-1)$-planes that intersect X.

$Z \subseteq Gr(n-m-1, \mathbb{P}^n)$, has codimension 1

then there exists a holomorphic $f \in H^0(Gr, \mathcal{O}(d))$ which vanishes precisely on Z. Now we're in the hypersurface situation

f defines a point in $\mathbb{P}(H^0(Gr, \mathcal{O}(d)))$, which we call the chow point, ~~which~~ denoted $\mathrm{Chow}(X)$.

chow stability $\longleftrightarrow$ Chow point

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more generally (X, L^k) embeds in $\mathbb{P}^{N_k}$

$Z \subseteq Gr(N_k - m - 1, \mathbb{P}^n)$ defines $\mathrm{Chow}_k(X)$

Asymptotic Chow stable $\Leftrightarrow$ chow stable for k large

Example Hypersurfaces $X \subset \mathbb{P}^N$, defined by $f \in H^0(\mathcal{O}(d))$

$$\dim_{\mathbb{C}} X = N-1, \text{ so } \mathbb{G}(N-(N-1)-1, N) = \mathbb{G}(0, N)$$

or 0-dim points in $\mathbb{P}^n$. the set of points which intersect X is just X. This f is the chow form

Example Points in $\mathbb{P}^n$. Let $p = [p_0 : p_1 : \dots : p_N] \in \mathbb{P}^N$

Z_p is the set of hyperplanes in $\mathbb{P}^N$ which contain p.

$$L \in H^0(\mathcal{O}(1)) \triangleq a_0 z_0 + a_1 z_1 + \dots + a_N z_N = 0$$

L contains $L(p) = 0 \Rightarrow a_0 p_0 + \dots + a_N p_N = 0$

Z_p is defined by vanishing of $\cancel{\ }$ homo deg 1

$$f_p = p(L) = p_0 a_0 + \dots + p_N a_N$$ ← polynomial in a_i

given multiple points, $q_1, \dots, q_k$

Z_p defined by vanishing of $f = f_{q_1} f_{q_2} \cdots f_{q_k}$

Symplectic Reduction

$$\begin{array}{ccccc} K & < & G & \circlearrowright & X \\ \downarrow & & \downarrow & & \cap \\ SU(n+1) & \subset & SL(n+1,\mathbb{C}) & \circlearrowright & \mathbb{P}^n \end{array}$$

K preserves $\omega = \omega_{FS}$, acts by symplectomorphism

Let $\tilde{V} \in TX$ preserve ω. $\tilde{V}$ defined by $V \in k$

$$\mathcal{L}_{\tilde{V}} \overset{0}{\omega} = d(\omega(\tilde{V},\cdot)) + \cancel{d\omega}^{0}(\tilde{V},\cdot,\cdot)$$

$\Rightarrow \quad \omega(\tilde{V},\cdot) \in H^1(X,\mathbb{R})$. (Action given by $\mathbb{P}^n$ simply connected)

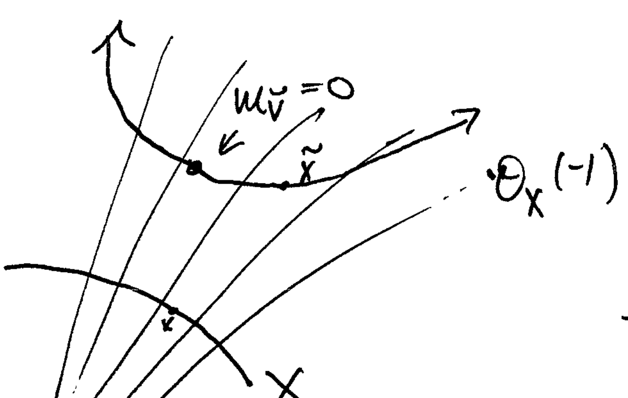
$\Rightarrow \quad \omega(\tilde{V},\cdot) = dm_{\tilde{V}}$

then moment map $m : X \longrightarrow k^*$

defined by $\langle m(x), v\rangle = m_{\tilde{V}}(x)$

For ~~us[illegible]~~ work upstairs "morally"

lift any x to $\tilde{x}$, then



$$\omega = dd^c \log\|\tilde{x}\|$$

now $dd^c \log\|\tilde{x}\|(\tilde{V},\cdot) = dm_{\tilde{V}}$

thus $d^c \log\|\tilde{x}\|(\tilde{V}) = m_{\tilde{V}}$

so $m_{\tilde{V}} = J\tilde{V} \log\|\tilde{x}\|$

makes since because $\tilde{V}$ preserves the metric

if $\lambda(t) \in U(1) \subset \mathbb{C}^* \Rightarrow \frac{d}{dt} \log \|\tilde{x}\| = 0$

yet in the <u>radial</u> direction (same as by $J\tilde{V}$ for $V \in k$)

$$m_V = \frac{d}{dt}\Big|_{t=1} \log \| \lambda . \tilde{x} \|_{t \in (0,\infty)}$$

furthermore
$$J\tilde{V}(m_{\tilde{V}}) = dm_{\tilde{V}}(J\tilde{V}) = \omega(\tilde{V}, J\tilde{V})$$
$$= g(\tilde{V}, \tilde{V}) \geq 0$$

thus
$$\frac{d}{dt}^2\Big|_{t=1} \log \| \lambda \cdot \tilde{x} \|_{t \in (0,\infty)} \geq 0$$
convex !

Thus, the orbit tends to ∞ at both ends (i.e. orbit is <u>closed</u>)

$\Leftrightarrow$ has a critical point of $\log \| \lambda \cdot \tilde{x} \|$ where $m_{\tilde{V}} = 0$!!!

This modivates

<u>Thm</u> [Kempf - Ness]

$$X_{ss}/G \cong m^{-1}(0)/K$$

don't need to consider the whole G action since all closed orbits have zeros of the moment map

Moment map for $SL(2,\mathbb{C}) \supset SU(2) \circlearrowright \mathbb{P}^1$

$$e^{\tilde{v}} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in SU(2)$$

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \leftarrow \text{in complexified direction}$$

$$\| \lambda(t) \tilde{x} \|^2 = t \|\tilde{x}_1\|^2 + t^{-1} \|\tilde{x}_2\|^2$$

$$\frac{\partial}{\partial t} \log \| \lambda(t) \tilde{x} \|^2 = \frac{\|\tilde{x}_1\|^2 - \frac{1}{t^2} \|\tilde{x}_2\|^2}{t \|\tilde{x}_1\|^2 + t^{-1} \|x_2\|^2}$$

$$\frac{\partial}{\partial t}\Big|_{t=0} \log \| \lambda(t) \tilde{x} \|^2 = \frac{\|\tilde{x}_1\|^2 - \|\tilde{x}_2\|^2}{\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2} = m_{\tilde{v}}$$

So if $X \in \mathbb{P}^1$, $X = [\tilde{x}_1 : \tilde{x}_2] = [\frac{\tilde{x}_1}{\tilde{x}_2} : 1]$

$$\Rightarrow \quad m_{\tilde{v}} = \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \quad \leftarrow \text{ z - coordinate in stereographic projection}$$

$su(2)$ is 3 dim, prob get other 2 coordinates similarly

thus $m : \mathbb{P}^1 \to su(2)^*$ is just inclusion into unit sphere!

Thus given n points in $\mathbb{P}^1$

moment map is $m = \sum_{i=1}^{n} m_i$

vector sum of points in $\mathbb{R}^3$. $m^{-1}(0)$ is set of "balanced configurations", center of mass $0 \in \mathbb{R}^3$

Thus

<u>Thm</u> a configuration of points with multiplicities on $S^2 \subset \mathbb{R}^3$ can be moved by an element of $SL(2,\mathbb{C})$ to have center of mass at O

$\iff$ f ~~$\in H^0(\mathcal{O}(n))$~~ is stable

$f \in \mathbb{C}[x,y]$ does not vanish of order $> \frac{n}{2}$

each multiplicity is strictly less than half total.

A version of this thm for Chow stability

<u>Thm</u> (Zhang, Phong, Sturm)

$X \subseteq \mathbb{CP}^N$ a smooth projective variety

Then Chow(X) is stable

$\iff$ $\exists$ a unique σ_0 in $SU(N+1)\backslash SL(N+1,\mathbb{C})$ s.t. $\sigma_0(X)$ is balanced

$$\frac{1}{\mathrm{Vol}(X)} \int_{\sigma_0(X)} \left(\frac{\bar{z}_j z_i}{|z_0|^2 + \cdots + |z_N|^2} \right) \omega_{FS}^n = \frac{1}{N+1} \delta_{ij}$$

(6) $\frac{d^2}{dt} F^0_\omega (\varphi_{\sigma(t)}) \geq 0$

Now once again

If $\dim_{\mathbb{C}} X = n$, $\deg X = d$

$$\mathrm{Chow}(X) \triangleq \{ z \in Gr(N-n-1, \mathbb{P}^N) \mid z \cap X \neq \emptyset \}$$

also $Pl : Gr(N-n-1, \mathbb{P}^N) \longrightarrow \mathbb{P}^m$

Plucker embedding

let $\omega_{Gr} = Pl^* \Omega_{FS}$

If $f \in H^0(Gr, \mathcal{O}(d))$

$$\log \|f\|^2 = \frac{1}{D} \int_{Gr} \log \frac{|f(z)|^2}{|Pl(z)|^{2d}} \omega_{Gr}^{(N-n)(n+1)}$$

volume of Gr

Thm (zhang)

$$F^0_\omega(\varphi_\sigma) = -\frac{1}{V(n+1)} \log \frac{\|\sigma \cdot \mathrm{Chow}(X)\|^2}{\|\mathrm{Chow}(X)\|}$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} \frac{1}{V(n+1)} \log \frac{\|\sigma \,\mathrm{Chow}(X)\|^2}{\|\mathrm{Chow}(X)\|} = \frac{1}{V} \int_X \frac{x^* \bar{\sigma}^* (c^* + c) \sigma x}{x^* \bar{\sigma} \sigma x} \omega_\varphi^n$$

moment map!

let $\sigma(X)$ have coordinates $[Z_0 : \cdots : Z_N]$

Then $$\int_{\sigma(X)} \frac{Z^*(C^*+C)Z}{|Z|^2}\, \omega_{FS}^n = 0$$

for all C traceless iff

$$\int_{\sigma(X)} \frac{z_i \bar{z}_j}{|z|^2}\, \omega_{FS}^n = \frac{\mathrm{vol}(X)}{N+1}\, \delta_{ij}$$

Can do similar "balancing" act for Hermitian V.B and Gieseker-stability.

Mabuchi energy?

Note this links stability, bdd below. Valentino's paper

Lets do K-stability

Let (X, L) be a polarised variety with hilbert polynomial $P(k) := \chi(L^{\otimes k})$

for k large enough $\chi(L^{\otimes k}) = h^0(X, L^{\otimes k})$

A Test configuration for (X, L) consists of

① A flat projective family $\mathcal{L} \curvearrowright \mathcal{X} \to \mathbb{C}$

(all fibers $(X_t, \mathcal{L}_t)$ have same hilbert polynomial)

② $\mathbb{C}^*$ action, st $(\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t})$ is isomorphic to (X, L) for $t \in \mathbb{C} \setminus \{0\}$

Since 0 is fixed, we get induced action on $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$ and hence $H^0(\mathcal{X}_0, \mathcal{L}^k|_{\mathcal{X}_0})$. Let $w(k)$ be total weight of the action

$$P(k) = a_0 k^n + a_1 k^{n-1} + \dots \qquad w(k) = b_0 k^{n+1} + b_1 k^n + \dots$$

Let $F_1 = \dfrac{b_0 a_1 - b_1 a_0}{a_0^2}$ (reduces to usual definition of Futaki if X_0 is smooth)

K stable if $F_1 > 0$ (≥ 0)
semi

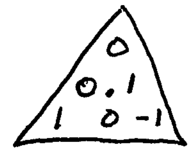
for all non-trivial test configurations

<u>Example</u> curve in $\mathbb{P}^2$ $\quad X \triangleq \{f=0\}$

as we saw, $f = a_3 xz + a_4 y^2 + a_6 z^2$ is smooth

look at $\quad xz + y^2 - z^2$

chow semi-stable

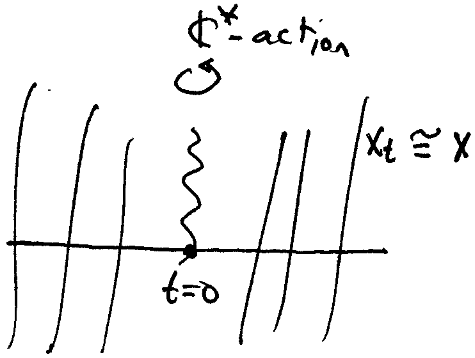


lets look at K-stability

consider 1-PS $\lambda(t) \to \begin{pmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{pmatrix}$ $\quad$ ~~a+b+c=0~~

$$\lambda(t)\cdot f = txz + t^{-2}y^2 - t^{-2}z^2$$

consider the ideal $\mathcal{I} = (txz + t^{-2}y^2 - t^{-2}z^2) \subseteq \mathbb{C}[t,t^{-1},x,y]$

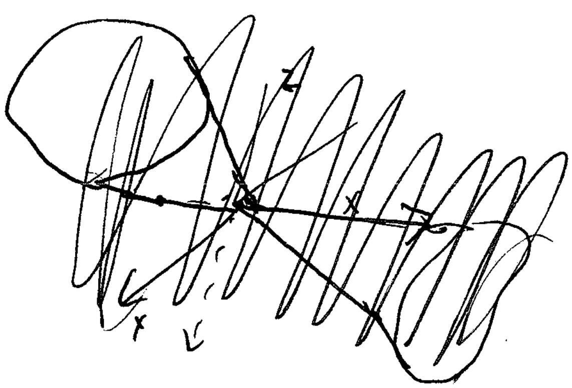


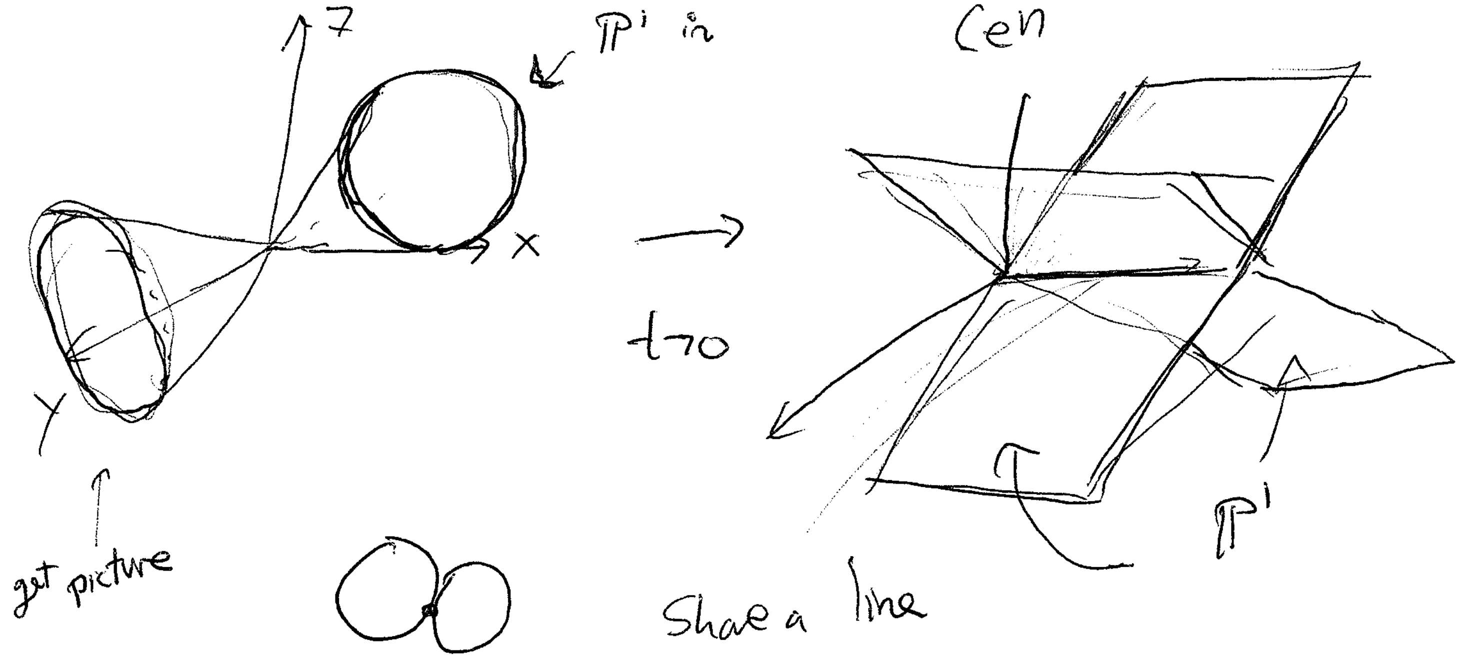
look at $\quad \mathcal{I} \cap \mathbb{C}[t,x,y] \subseteq \mathbb{C}[t,t^{-1},x,y]$

given by $\quad t^3 xz + y^2 - z^2$

now take t to zero $\quad$ get $y^2 - z^2$

central fiber





What is hilbert polynomial for K large

$$P(K) = h^0\left(X, \mathcal{O}_{\mathbb{P}^2}(K)\Big|_X\right)$$

since X is $\mathbb{P}^1$, deg 2 curve, really just

$$h^0(\mathcal{O}_{\mathbb{P}^1}(2K)) = 2K + 1$$

Lets compute the total weight

central fibe

$$H^0(X_0, \mathcal{O}_{\mathbb{P}^2}(K)\Big|_{X_0}) = \frac{\mathbb{C}[x,y,z]}{(y^2 - z^2)} = \{y^K, zy^{K-1}, xy^{K-1}, xzy^{K-2}, \dots, x^{K-1}y, x^{K-1}z, x^K\}$$

Now λ acts the same on y and z,

so really ~~the~~ the weight on y^K and zy^{K-1} is $\boxed{-K}$

on xy^{K-1} and xzy^{K-2} is $-K+3$

total weight (add them up) is

$$w(K) = 2\left(\sum_{m=0}^{K-1} (-K+3m)\right) + 2K \qquad \left(S_n = \overset{\text{number of terms}}{K}\,\underset{\text{last - first}}{(K-1)}\right.$$

$$= 2\left(-K^2 + \frac{3}{2}(K^2 - K)\right) + 2K$$

$$= -2K^2 + 3K^2 - 3K + 2K$$

$$= K^2 - K$$

Thus $b_0 = 1 \qquad b_1 = -1$

$\qquad a_0 = 2 \qquad a_1 = 1$

So $$F_1 = \frac{(1)(1) - (-1)(2)}{4} = \frac{3}{4} > 0$$

which is good, since

Asymptotically Chow semi-stable $\Rightarrow$ K-semi-stable

Now Slope Stability

given (X,L), $P(k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$

Define $\mu(X) = \frac{a_1}{a_0}$ $\sim \mu(X,L)$

Let Z be arbitrary sub scheme of X. $\pi : \hat{X} \to X$ blowup $\pi^{-1}(Z) = E$

define $\varepsilon(Z) = \varepsilon(Z, X, L)$.

$= \sup \{ c \mid L^k \otimes \mathcal{I}_Z^{ck}$ is globally generated for $k >> \}$

$= \sup \{ c \mid L(-cE)$ is ample on $\hat{X} \}$

Ex point on $\mathbb{P}^1$ $L = \mathcal{O}(1)$

point p ~~[illegible]~~ divisor defined by $\mathcal{O}(-1)$

$\sup \{ c \mid \mathcal{O}(k) \otimes \mathcal{O}(-ck)$ globally g $\} = 1$

Also think $\pi : \hat{X} \to \mathbb{P}^2$ blowup of 1 point

given positive metric h, on $\mathcal{O}(1)$

~~[illegible]~~ consider, near E, we have ~~[illegible]~~

locally consider

$\eta \leftarrow$ pullback of fubini study metric on $\mathbb{P}^1$ under map $(z, \ell) \mapsto \ell$

so curv of $-\eta$ is positive

only defined locally so need bump function

does $\rho_1 h_1 + \rho_2$ have positive curvature?

No, since bump function. however curvature of $\rho\eta$ is bounded so $h - (\rho\eta)^c$ has pos. curv for c small enough.

$\Omega_{L(-E)} = \Omega_L + \Omega_{[-E]}$

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for fixed $x \in \mathbb{Q}$ define $a_i(x)$ by

$$\chi(L^k(-xkE)) = a_0(x)\, k^n + a_1(x)\, k^{n-1} + O(k^{n-2}) \quad k \gg 0, \; xk \in \mathbb{N}$$

$a_i(x)$ is polynomial in x of degree at most $n-i$

extend to $\mathbb{R}$

by Riemann-Roch

$$a_0(x) = \frac{1}{n!} \int_{\hat{X}} c_1(L(-xE))^n$$

$$a_1(x) = -\frac{1}{2(n-1)!} \int_{\hat{X}} c_1(K_{\hat{X}}) \wedge c_1(L(-xE))^{n-1}$$

Define

$$\mu_c(\mathcal{I}_Z) = \mu_c(\mathcal{I}_Z, L) = \frac{\int_0^c \left(a_1(x) + \frac{a_0'(x)}{2}\right) dx}{\int_0^c a_0(x)\, dx}$$

(X, L) is slope semistable wrt Z if

$$\mu_c(\mathcal{I}_Z) \leq \mu(X) \quad \forall\, c \in (0, \varepsilon(Z)]$$

Stable if $(<)$ and for $c = \varepsilon(Z)$

if $\varepsilon(Z)$ is rational then global sections of $L^k \otimes \mathcal{I}_Z^{\varepsilon(Z)k}$ generate $L(-\varepsilon(Z)kE)$ on $\hat{X}$

Now just like if $0 \to S \to E \to Q \to 0$ and $c_1(S) + c_1(Q) = c_1(E)$

$$\Rightarrow \mu(S) < \mu(E) \iff \mu(Q) > \mu(E)$$

we can define the quotient slope of Z (slightly diff)

$$\chi\left(L^k \otimes \mathcal{I}_Z^{xk} / \mathcal{I}_Z^{xk+1}\right) = \alpha_1(x) k^{n-1} + \alpha_2(x) k^{n-2} + O(k^{n-3})$$

$$\mu_c(\mathcal{O}_Z) = \frac{\int_0^c (c-x)\, \alpha_2(x)\, dx + \frac{c}{2} \alpha_1(0)}{\int_0^c (c-x)\, \alpha_1(x)\, dx}$$

[Note $\chi\left(L^k|_Z \otimes S^{xk} \nu_Z^*\right)$ gives same if Z smooth

$\nu_Z = \left(\frac{\mathcal{I}_Z}{\mathcal{I}_Z^2}\right)^*$]

Take $X \times \mathbb{C}$

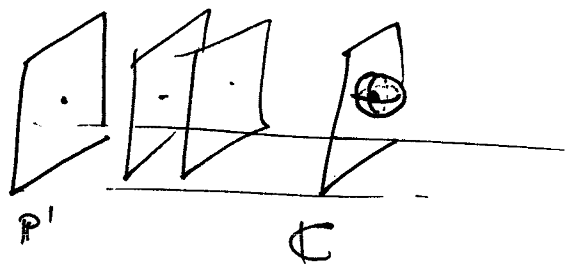
blow up $Z \times \{0\}$ denote exp divisor as

$\pi : \mathcal{X} \to X \times \mathbb{C}$ $\quad \pi^{-1}(Z \times \{0\}) = P$

Then central fiber is just

$\hat{X}$ glued to P along E, $\mathcal{X}_0 = \hat{X} \cup_E P$

Let $\hat{X}$ be $\mathbb{P}^1$ blown up at 1 point



$\mathbb{C}^* \curvearrowright \mathcal{X}$ by acting on t with weight -1

trivial away from central fiber

$$F_1 = \frac{1}{a_0^2}\left(b_0 a_1 - b_1 a_0\right)$$

$$= -\frac{b_0}{a_0}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right)$$

$$= -\frac{b_0}{a_0}\left(\frac{b_1}{b_0} - \mu(X)\right)$$

$\boxed{\text{Claim}}$ for this test config, $b_0 < 0$, $a_0 > 0$

$\Rightarrow$ and $\frac{b_1}{b_0} = \mu(\mathcal{O}_Z)$. so $F_1 \geq 0 \Rightarrow \mu(\mathcal{O}_Z) \geq \mu(X)$

Proof of Claim

let $\mathcal{L}_c = L(-cP)$ ample if c is rational

First we compute $H^0(\mathcal{X}, \mathcal{L}_c^k)$

$$= H^0(\mathcal{X}, (L(-cE))^k)$$

$$= H^0(X \times \mathbb{C}, L^k \otimes (\mathcal{I}_Z + (t))^{ck})$$

$$\bigoplus_{i=1}^{ck} t^{ck-i} H^0(X, L^k \otimes \mathcal{I}_Z^{i}) \oplus t^{ck} \mathbb{C}[t]\, H^0(L^k)$$

Now
$$t\, H^0(\mathcal{X}, \mathcal{L}_c^k) = \bigoplus_{i=1}^{ck} t^{ck-i+1} H^0(X, L^k \otimes \mathcal{I}_Z^{i})$$

$$(i \to i+1) \quad = \bigoplus_{i=0}^{ck-1} t^{ck-i} H^0(X, L^k \otimes \mathcal{I}_Z^{i+1})$$

so $H^0(\mathcal{L}_c^k|_{\mathcal{X}_0})$

$$= H^0(\mathcal{X}, \mathcal{L}_c^k) / t\, H^0(\mathcal{X}, \mathcal{L}_c^k)$$

$$= H^0(X, L^k \otimes \mathcal{I}_Z^{ck}) \oplus \bigoplus_{i=0}^{ck-1} t^{ck-i} H^0\left(L^k|_Z \otimes \left(\frac{\mathcal{I}_Z^i}{\mathcal{I}_Z^{i+1}}\right)\right)$$

$$\frac{\mathcal{I}_Z^i}{\mathcal{I}_Z^{i+1}} = S^i \nu_Z^*$$

$\mathbb{C}^*$ acts with weight -1 on t

so
$$w(k) = -\sum_{i=0}^{ck-1} (ck-i)\, h^0(L^{\otimes k}|_Z \otimes S^i \nu_Z^*)$$

... go to paper...